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Ullemar's formula for the moment map, II ☆

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Abstract

We prove the complex analogue of Ullemar's formula for the Jacobian of the complex moment mapping. This formula was previously established in the real case.

Keywords: Moment map; Resultants; Jacobian; Univalent polynomials

1. Introduction

Consider the 'moment map'

$$\mu : \Omega \to (\mu_0, \mu_1, \mu_2, \ldots),$$

where

$$\mu_k = \frac{\mathrm{i}}{2\pi} \iint_\Omega \zeta^k \, \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}, \tag{1}$$

and Ω is a bounded domain in $\mathbb{C}$. If Ω is a simply-connected domain, we can uniformize it as the image $\Omega = \phi(\mathbb{D})$, where ϕ is a unique function which is holomorphic in the unit disk $\mathbb{D}$ and normalized by the following conditions:

$$\phi(0) = 0, \qquad \phi'(0) > 0. \tag{2}$$

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Then (1) takes the form

$$\mu_k(\phi) = \frac{\mathrm{i}}{2\pi} \iint\limits_{\mathbb{D}} \phi^k(z)|\phi'(z)|^2 \,\mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{3}$$

In general, when the function ϕ is not globally univalent in $\mathbb{D}$ and satisfies (2), we use the previous formula as the definition for the complex (or analytic) moments of ϕ [2]; then Ω is regarded as a Riemannian surface over $\mathbb{C}$.

This notion appears in several problems of complex analysis and its applications. In particular, the sequence $(\mu_k)_{k\geqslant 1}$ constitutes an infinite family of invariants of the Hele–Shaw problem [4] of the cell Ω and can be used as a canonic coordinate system in the corresponding Laplacian growth model [3] (see also [6] for the functional analysis interpretation).

In what follows, we consider the special case when the moment map μ is restricted to the set $\mathfrak{S}_n \subset \mathbb{C}[z]$ of polynomials of degree n normalized by (2). It is easy to verify that $\mu_k = 0$ for all $k \geqslant n$ (see also (13)), so only the first n moments are of interest for $P \in \mathfrak{S}_n$. We consider the induced finite dimensional map

$$\mu : P = a_1 z + a_2 z^2 + \cdots + a_n z^n \to (\mu_0(P), \ldots, \mu_{n-1}(P)). \tag{4}$$

Since $\mu_0(P) > 0$,

$$\mu : \mathfrak{S}_n \to \mathbb{R}^+ \times \mathbb{C}^{n-1}.$$

Notice that

$$\dim_{\mathbb{R}} \mathbb{R}^+ \times \mathbb{C}^{n-1} = \dim_{\mathbb{R}} \mathfrak{S}_n = 2n - 1,$$

where $\mathfrak{S}_n$ is regarded as an open subset of a linear space. In [2] Gustafsson proved that the Fréchet derivative $\mathrm{d}\mu$ is non-singular at any P which is a locally univalent polynomial.

On the other hand, notice that the subset $\mathfrak{S}_n^{\mathbb{R}}$ which consists of the polynomials with real coefficients is an invariant set of μ in the sense that all $\mu_k(P)$ are real (cf. (14)). Hence, in a similar way the moment map induces the map $\mu_{\mathbb{R}} : \mathfrak{S}_n^{\mathbb{R}} \to \mathbb{R}^n$.

In [7], Ullemar conjectured the following formula for the Jacobian of $\mu_{\mathbb{R}}$:

$$J_{\mathbb{R}}(P) := \frac{\partial(\mu_0, \ldots, \mu_{n-1})}{\partial(a_1, \ldots, a_n)}(P) = 2^{-\frac{n(n-3)}{2}} a_1^{\frac{n(n-1)}{2}} P'(1)P'(-1)\Delta(P'^*(z)), \tag{5}$$

where Δ stands for the principal Hurwitz determinant [1, §15.715], and Q^* denotes the mirror conjugate image of polynomial Q, i.e.

$$Q^*(z) := z^m \bar{Q}(1/z) = \bar{q}_m + \bar{q}_{m-1} z + \cdots + \bar{q}_0 z^m, \quad m = \deg Q, \tag{6}$$

where $\bar{Q}(z) = \overline{Q(\bar{z})}$ is the conjugate polynomial.

The above expression for the Jacobian was recently proved in [5] as a consequence of the following identity:

$$J_{\mathbb{R}}^2(P) = 4(-1)^{n-1} a_1^{n(n-1)} \mathscr{R}(P', P'^*) \cdot P'(-1)P'(1), \tag{7}$$

where $\mathscr{R}(\cdot,\cdot)$ denotes the resultant of the corresponding polynomials.

In this paper we generalize formula (7) for polynomials with arbitrary complex coefficients.

Theorem 1. *The Jacobian of the moment map is expressed as follows*:

$$J_{\mathbb{C}}(P) := \frac{\partial(\bar{\mu}_{n-1}, \ldots, \bar{\mu}_1, \mu_0, \mu_1, \ldots, \mu_{n-1})}{\partial(\bar{a}_n, \ldots, \bar{a}_2, a_1, a_2, \ldots, a_n)} = 2a_1^{n^2-n+1}\mathscr{R}(P', P'^{*}). \quad (8)$$

A well known theorem of Sylvester (see Section 2.2) allows us to compute the above resultant as the determinant of a matrix of size $2n-2$, whose entries are 0 or a coefficient of either P' or P'^{*}. In particular, the resultant is homogeneous in the coefficients of P' and P'^{*} separately, with respective degree $n-1$.

On the other hand, geometrically, the hypersurface

$$\{(a, \bar{a}) : \mathscr{R}(P', P'^{*}) = 0\},$$

i.e. the critical set of the Jacobian, is the projection of the incidence variety

$$\left\{(a, \bar{a}, z) : \sum_{k=1}^{n} k a_k z^{k-1} = \sum_{k=0}^{n-1} (n-k)\bar{a}_{n-k} z^k \right\},$$

that is to say, the set of $(a, \bar{a})$ which appear above for some z.

The following assertion is a direct consequence of the definition (15) of the resultant and formula (6) above, and it characterizes the set of critical points of $\mathrm{d}\mu$.

Corollary 2. *The moment map is degenerate at P if and only if the derivative P' has two roots α_i and α_j such that $\alpha_i\bar{\alpha}_j = 1$ (the case $i = j$ is permitted).*

Note that for a locally univalent polynomial in the closed unit disk we have $|\alpha_j| < 1$ for all the roots of its derivative. Hence, we obtain another proof of the above result due to Gustafsson [2].

Corollary 3. *The moment map is locally injective on the set of all locally univalent polynomials in the closed unit disk.*

2. Preliminaries

2.1. Complex moments

Using the Stokes formula, we obtain

$$\mu_k = \frac{\mathrm{i}}{2\pi(k+1)} \int_{\partial\Omega} \zeta^{k+1}\, \mathrm{d}\bar{\zeta} = \frac{1}{2\pi\mathrm{i}} \int_{\partial\Omega} \zeta^k \bar{\zeta}\, \mathrm{d}\zeta, \quad (9)$$

which implies

$$\mu_k(\phi) = \frac{\mathrm{i}}{2\pi(k+1)}\int_{\mathbb{T}} \phi^{k+1}(z)\bar{\phi}'(\bar{z})\,\mathrm{d}\bar{z} = \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}} \phi^{k}(z)\bar{\phi}(\bar{z})\phi'(z)\,\mathrm{d}z,$$

where $\mathbb{T} = \partial\mathbb{D}$ is the unit circle. Hence, using the identity $\bar{z} = 1/z$ which holds everywhere in $\mathbb{T}$, we get

$$\begin{aligned}\mu_k(\phi) &= \frac{1}{2\pi\mathrm{i}(k+1)}\int_{\mathbb{T}} \phi^{k+1}(z)\bar{\phi}'(1/z)\frac{\mathrm{d}z}{z^2}\\ &= \frac{1}{2\pi\mathrm{i}}\int_{\mathbb{T}} \phi^{k}(z)\bar{\phi}(1/z)\phi'(z)\,\mathrm{d}z.\end{aligned} \tag{10}$$

Given a function which is analytic in a neighborhood of $\mathbb{T}$, let us denote by $\lambda_s(f)$ the sth Laurent coefficient of f, i.e.

$$f(z) = \sum_{s=-\infty}^{\infty} \lambda_s(f) z^s,$$

hence

$$\mu_k(\phi) = \frac{1}{k+1}\lambda_1(\phi^{k+1}(z)\bar{\phi}'(1/z)) = \lambda_{-1}(\phi^k(z)\phi'(z)\bar{\phi}(1/z)). \tag{11}$$

Now, let P be an arbitrary polynomial in $\mathfrak{S}_n$. Then $\bar{P}'(1/z) = P^{'*}(z)z^{1-n}$ and $\bar{P}(1/z) = P^*(z)z^{-n}$, which by virtue of (11) yields

$$\mu_k(P) = \frac{1}{k+1}\lambda_n(P^{k+1}P^{'*}) = \lambda_{n-1}(P'P^kP^*). \tag{12}$$

It follows from the first identity in (12) and $P(0) = 0$ that

$$\mu_k(P) = 0, \quad k \geqslant n. \tag{13}$$

On the other hand, the second identity in (12) yields the so-called *Richardson formula*

$$\mu_k(P) = \sum s_1 a_{s_1}\cdots a_{s_{k+1}}\bar{a}_{s_1+\cdots+s_{k+1}}, \tag{14}$$

where the sum is taken over all possible sets of indices $s_1, \ldots, s_k \geqslant 1$. It is assumed that $a_j = 0$ for $j \geqslant n+1$. These formulae are easy to use for straightforward manipulations with the complex moments and it follows also that $\mu_k(P)$ is a *polynomial* mapping.

It is convenient to identify $\mathfrak{S}_n$ with the corresponding coefficient subset in $\mathbb{R}^+ \times \mathbb{C}^{n-1}$ in a standard way:

$$a \sim P := a_1 z + a_2 z^2 + \cdots + a_n z^n.$$

Since,

$$\mu_0(P) = \sum_{s=1}^{n} s|a_s|^2 > 0, \quad \mu_{n-1}(P) = na_1^n \bar{a}_n \neq 0,$$

the moment map (4) is well defined as an automorphism of $\mathfrak{S}_n$ into itself.

2.2. *Resultants*

Here we review some basic facts about the resultant; see [8] for a detailed introduction.

The *resultant* of two polynomials

$$A(z) = a_m \prod_{j=1}^{m} (z - \alpha_j), \quad B(z) = b_k \prod_{j=1}^{k} (z - \beta_j)$$

with respect to z is the polynomial

$$\mathscr{R}(A, B) = a_m^k b_k^m \prod_{i,j=1} (\alpha_i - \beta_j). \tag{15}$$

The resultant vanishes iff A and B have a common root. It can be evaluated as the determinant of the *Sylvester matrix*, which is the following $m + k$ by $m + k$ matrix:

$$\begin{pmatrix} a_0 & a_1 & \dots & \dots & a_m & & & & \\ & a_0 & a_1 & \dots & \dots & a_m & & & \\ & & & & \vdots & & & & \\ & & & a_0 & a_1 & \dots & \dots & a_m & \\ b_0 & b_1 & \dots & b_k & & & & & \\ & b_0 & b_1 & \dots & b_k & & & & \\ & & & & \vdots & & & & \\ & & & b_0 & b_1 & \dots & \dots & b_k & \end{pmatrix}$$

in which the first k rows are the coefficients of A, the next m rows are the coefficients of B, and the elements not shown are all zero. The following are some useful elementary properties we will use below.

$$\begin{aligned} &\mathscr{R}(A, B) = (-1)^{km} \mathscr{R}(B, A), \\ &\mathscr{R}(A_1 A_2, B) = \mathscr{R}(A_1, B)\mathscr{R}(A_2, B), \\ &\mathscr{R}(z^n, A) = A^n(0). \end{aligned} \tag{16}$$

Next, given a polynomial $A(z)$ of degree n, we define its mirror conjugate image as

$$A^*(z) := z^n \bar{A}(1/z) = \bar{a}_n + \bar{a}_{n-1} z + \cdots + \bar{a}_0 z^n,$$

where $\bar{A}(z) = \overline{A(\bar{z})}$ is the conjugate polynomial. We have for their roots: $\alpha_j^* = (\bar{\alpha_j})^{-1}$ and the corresponding resultant takes the following form:

$$\mathscr{R}(A, A^*) = \det \begin{pmatrix} a_0 & a_1 & \dots & \dots & a_n & & & \\ & a_0 & a_1 & \dots & \dots & a_n & & \\ & & & & \vdots & & & \\ & & & a_0 & a_1 & \dots & \dots & a_n \\ \bar{a}_n & \bar{a}_{n-1} & \dots & \dots & \bar{a}_0 & & & \\ & \bar{a}_n & \bar{a}_{n-1} & \dots & \dots & \bar{a}_0 & & \\ & & & & \vdots & & & \\ & & & \bar{a}_n & \bar{a}_{n-1} & \dots & \dots & \bar{a}_0 \end{pmatrix}. \tag{17}$$

Remark 4. We wish to point out that the latter form, $\mathscr{R}(A, A^*)$, is irreducible as a polynomial of $(a, \bar{a})$ over $\mathbb{C}$. The proof is given in [5, Theorem 6].

3. Proof of the Theorem

First, we evaluate the partial derivative of the moment map. Namely, we have for all $k = 0, \dots, n-1$, $j = 1, \dots, n$, except for $j = 1, k = 0$,

$$\begin{aligned} \frac{\partial \mu_k(P)}{\partial a_j} &= \lambda_{n-j}(P^{'*}P^k), \\ \frac{\partial \mu_k(P)}{\partial \bar{a}_j} &= \lambda_{j-1}(P'P^k). \end{aligned} \tag{18}$$

In fact, let j be an integer from $\{2, \dots, n\}$. Then by the first identity in (12) we have for

$$\frac{\partial \mu_k(P)}{\partial a_j} = \frac{1}{k+1}\lambda_n\left(P^{'*}\frac{\partial P^{k+1}}{\partial a_j}\right) = \lambda_n(P^{'*}P^k z^j) = \lambda_{n-j}(P^{'*}P^k).$$

Similarly, using $P^* = \bar{a}_n + \bar{a}_{n-1}z + \dots + \bar{a}_2 z^{n-2} + a_1 z^{n-1}$ and the second identity in (12) we obtain

$$\frac{\partial \mu_k(P)}{\partial \bar{a}_j} = \lambda_{n-1}\left(P'P^k\frac{\partial P^*}{\partial \bar{a}_j}\right) = \lambda_{n-1}(P'P^k z^{n-j}) = \lambda_{j-1}(P'P^k).$$

Finally, for $j = 1$ we have by the first identity in (12)

$$\begin{aligned} \frac{\partial \mu_k(P)}{\partial a_1} &= \frac{1}{k+1}\lambda_n\left(P^{'*}\frac{\partial P^{k+1}}{\partial a_1} + P^{k+1}\frac{\partial P^{'*}}{\partial a_1}\right) \\ &= \lambda_{n-k}(P^{'*}P^k) + \frac{1}{k+1}\lambda_1(P^{k+1}). \end{aligned}$$

But $\lambda_1(P^{k+1}) = 0$ for $k \geqslant 1$, hence the desired assertion follows.

We will make use of the following notation

$$\nabla f := \left(\frac{\partial f}{\partial a_n}, \frac{\partial f}{\partial a_{n-1}}, \ldots, \frac{\partial f}{\partial a_2}, \frac{\partial f}{\partial a_1}, \frac{\partial f}{\partial \bar{a}_2}, \ldots, \frac{\partial f}{\partial \bar{a}_{n-1}}, \frac{\partial f}{\partial \bar{a}_n}\right),$$

and by

$$q_{j-1} = ja_j$$

we denote the coefficients of the derivative $Q := P'$. Then

$$\nabla \mu_0 = (\bar{q}_{n-1}, \ldots, \bar{q}_1, 2q_0, q_1, \ldots, q_{n-1}), \tag{19}$$

and for all $k = 1, \ldots, n-1$ we have from (18)

$$\begin{aligned}\nabla \mu_k = \big(&\lambda_0(Q^* P^k), \ldots, \lambda_{n-2}(Q^* P^k), \lambda_{n-1}(Q^* P^k),\\ &\lambda_1(QP^k), \ldots, \lambda_{n-1}(Q^* P^k)\big).\end{aligned} \tag{20}$$

Let $\mathbf{Y}_0 = \nabla \mu_0$ and for $k \geqslant 1$ write

$$\begin{aligned}\mathbf{Y}_k := \big(&\lambda_0(Q^* z^k), \ldots, \lambda_{n-2}(Q^* z^k), \lambda_{n-1}(Q^* z^k),\\ &\lambda_1(Qz^k), \ldots, \lambda_{n-1}(Q^* z^k)\big).\end{aligned}$$

As a direct consequence of the above formula we conclude that

$$\mathbf{Y}_k = \mathbf{0}, \quad k \geqslant n.$$

Then it follows from (20) and

$$P = z(a_1 + \cdots + a_n z^{n-1})$$

that for all $k \geqslant 1$

$$\nabla \mu_k = a_1^k \mathbf{Y}_k + \sum_{j=k+1}^{n-1} w_{k,j} \mathbf{Y}_j.$$

Thus,

$$\nabla \mu_0 \wedge \nabla \mu_1 \wedge \cdots \wedge \nabla \mu_{n-1} = a_1^N \mathbf{Y}_0 \wedge \mathbf{Y}_1 \wedge \cdots \wedge \mathbf{Y}_{n-1}, \tag{21}$$

where $N = (n-1)n/2$.

On the other hand, for all $k \geqslant 1$ we have

$$\mathbf{Y}_k := (0, \ldots, 0, \bar{q}_{n-1}, \ldots, \bar{q}_k, 0, \ldots, 0, q_0, q_1, \ldots, q_{k-1}),$$

where the zeroes groups contain k and $k-1$ items respectively.

Now we treat the conjugate moments. We have $\bar{\mu}_k(P) = \mu_k(\overline{P})$, whence

$$\nabla \bar{\mu}_k = (\nabla \mu_k)^*,$$

where by $\mathbf{X}^*$ we denote the mirror conjugate image of vector $\mathbf{X} = (x_1, x_2, \ldots, x_{2n-1})$, i.e.

$$\mathbf{X}^* = (\bar{x}_{2n-1}, \ldots, \bar{x}_2, \bar{x}_1).$$

Repeating the above argument for the conjugate expressions yields

$$\nabla\bar{\mu}_{n-1} \wedge \nabla\bar{\mu}_{n-2} \wedge \cdots \wedge \nabla\bar{\mu}_1 = a_1^N \mathbf{Y}^*_{n-1} \wedge \mathbf{Y}^*_{n-2} \wedge \cdots \wedge \mathbf{Y}^*_1, \tag{22}$$

hence

$$\begin{aligned}&\nabla\bar{\mu}_{n-1} \wedge \cdots \wedge \nabla\bar{\mu}_1 \wedge \nabla\mu_0 \wedge \nabla\mu_1 \wedge \cdots \wedge \nabla\mu_{n-1}\\ &\quad = a_1^{n^2-n} \mathbf{Y}^*_{n-1} \wedge \cdots \wedge \mathbf{Y}^*_1 \wedge \mathbf{Y}_0 \wedge \mathbf{Y}_1 \wedge \cdots \wedge \mathbf{Y}_{n-1}.\end{aligned} \tag{23}$$

We rewrite the latter identity in terms of determinants which gives the following expression for the Jacobian:

$$J_{\mathbb{C}}(P) = \frac{\partial(\bar{\mu}_{n-1}, \ldots, \bar{\mu}_1, \mu_0, \mu_1, \ldots, \mu_{n-1})}{\partial(\bar{a}_{n-1}, \ldots, \bar{a}_1, a_0, a_1, \ldots, a_{n-1})} = a_1^{n^2-n} \det \mathbf{Y}, \tag{24}$$

where

$$\mathbf{Y} = \left(\begin{array}{cccccccccc}
q_0 & & & & q_{n-1} & & & & & \\
\bar{q}_1 & q_0 & & & q_{n-2} & q_{n-1} & & & & \\
\vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & & & \\
\bar{q}_{n-2} & \bar{q}_{n-3} & \cdots & \bar{q}_0 & q_1 & q_2 & q_3 & \cdots & q_{n-1} & \\
\bar{q}_{n-1} & \bar{q}_{n-2} & \cdots & \bar{q}_1 & 2q_0 & q_1 & q_2 & \cdots & q_{n-2} & q_{n-1} \\
 & \bar{q}_{n-1} & \cdots & \bar{q}_2 & \bar{q}_1 & q_0 & q_1 & \cdots & q_{n-3} & q_{n-2} \\
 & & \ddots & \vdots & \vdots & & \ddots & & \vdots & \vdots \\
 & & & \bar{q}_{n-1} & \bar{q}_{n-2} & & & & q_0 & q_1 \\
 & & & & \bar{q}_{n-1} & & & & & q_0
\end{array}\right),$$

and the elements not shown are all zero.

Now, let $\mathbf{X}_j$ denote the jth column in $\mathbf{Y}$. We have for $j = 1, \ldots, n-1$

$$\mathbf{X}_j = (0, \ldots, 0, q_0, \bar{q}_1, \ldots, \bar{q}_{n-1}, 0, \ldots, 0)^{\mathrm{T}},$$

with $j-1$ first zeroes, and for $j = n+1, \ldots, 2n-1$:

$$\mathbf{X}_j = (0, \ldots, 0, q_{n-1}, \ldots, q_1, q_0, 0, \ldots, 0)^{\mathrm{T}},$$

with $j-n$ first zeroes, and

$$\mathbf{X}_n = (q_{n-1}, \ldots, q_1, 2q_0, \bar{q}_1, \ldots, \bar{q}_{n-1})^{\mathrm{T}}.$$

One can readily verify that

$$\mathbf{X}_n - \sum_{j=1}^{n-1} \frac{q_{n-j}}{q_0}\mathbf{X}_j + \sum_{j=n+1}^{2n-1} \frac{\overline{q}_{j-n}}{q_0}\mathbf{X}_j = (0, \ldots, 0, 2q_0, 2\bar{q}_1, \ldots, 2\bar{q}_{n-1})^{\mathrm{T}},$$

which yields for the determinant

$$\det \mathbf{Y} = 2\det \begin{pmatrix} q_0 & & & & & & & & & \\ \bar{q}_1 & q_0 & & & & q_{n-1} & & & & \\ \vdots & \vdots & \ddots & & & \vdots & & \ddots & & \\ \bar{q}_{n-1} & \bar{q}_{n-2} & \cdots & \bar{q}_1 & q_0 & q_1 & q_2 & \cdots & q_{n-2} & q_{n-1} \\ & \bar{q}_{n-1} & \cdots & \bar{q}_2 & \bar{q}_1 & q_0 & q_1 & \cdots & q_{n-3} & q_{n-2} \\ & & \ddots & \vdots & \vdots & & \ddots & & \vdots & \vdots \\ & & & \bar{q}_{n-1} & \bar{q}_{n-2} & & & & q_0 & q_1 \\ & & & & \bar{q}_{n-1} & & & & & q_0 \end{pmatrix}. \tag{25}$$

The latter is the transposed Sylvester matrix of $Q^*(z)$ and $zQ(z)$, hence by (16)

$$\begin{aligned} \det \mathbf{Y} &= 2\mathscr{R}(Q^*, zQ) = 2(-1)^{n(n-1)}\mathscr{R}(zQ, Q^*) \\ &= 2\mathscr{R}(z, Q^*)\mathscr{R}(Q, Q^*) = 2Q(0)\mathscr{R}(Q, Q^*). \end{aligned} \tag{26}$$

Thus, using our notation $Q = P'$ we arrive at

$$J_{\mathbb{C}}(P) = 2a_1^{n^2-n+1}\mathscr{R}(P', P'^*),$$

which completes the proof. □

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